Chapter 19

Examination-Type Questions

19.1 Dynamical Systems with Applications

Typically, students would be required to answer five out of eight questions in three hours. The examination would take place without access to a mathematical package. The student will require a calculator and graph paper.

1. (a) Sketch a phase portrait for the following system showing all isoclines:

$$\frac{dx}{dt} = 3x + 2y, \quad \frac{dy}{dt} = x - 2y.$$

[6]

(b) Show that the system

$$\frac{dx}{dt} = xy - x^2y + y^3, \quad \frac{dy}{dt} = y^2 + x^3 - xy^2$$

can be transformed into

$$\frac{dr}{dt} = r^2\sin(\theta), \quad \frac{d\theta}{dt} = r^2\left(\cos(\theta) - \sin(\theta)\right)\left(\cos(\theta) + \sin(\theta)\right)$$

using the relations $r\dot{r} = x\dot{x} + y\dot{y}$ and $r^2\dot{\theta} = x\dot{y} - y\dot{x}$. Sketch a phase portrait for this system given that there is one nonhyperbolic critical point at the origin.

[14]

2. (a) Prove that the origin of the system

$$\frac{dx}{dt} = -\frac{x}{2} + 2x^2y, \quad \frac{dy}{dt} = x - y - x^3$$

is asymptotically stable using the Lyapunov function $V = x^2 + 2y^2$.

[6]

(b) Solve the differential equations

$$\frac{dr}{dt} = -r^2, \quad \frac{d\theta}{dt} = 1,$$

given that $r(0) = 1$ and $\theta(0) = 0$. Hence show that the return map, say, $\mathbf{P}$, mapping points, say, r_n, on the positive x-axis to itself is given by

$$r_{n+1} = \mathbf{P}(r_n) = \frac{r_n}{1 + 2\pi r_n}.$$

[14]

3. (a) Find the eigenvalues of the following system and sketch a phase portrait in 3-dimensional space

$$\frac{dx}{dt} = -2x - z, \quad \frac{dy}{dt} = -y, \quad \frac{dz}{dt} = x - 2z.$$

[12]

(b) Show that the origin of the following nonlinear system is not hyperbolic:

$$\frac{dx}{dt} = -2y + yz, \quad \frac{dy}{dt} = x - xz - y^3, \quad \frac{dz}{dt} = xy - z^3.$$

Prove that the origin is asymptotically stable using the Lyapunov function $V = x^2 + 2y^2 + z^2$. What does asymptotic stability imply for a trajectory $\gamma(t)$ close to the origin?

[8]

4. (a) Consider the 2-dimensional system

$$\frac{dr}{dt} = r(\mu - r)\left(\mu - r^2\right), \quad \frac{d\theta}{dt} = -1.$$

Show how the phase portrait changes as the parameter μ varies and draw a bifurcation diagram.

[10]

(b) Prove that none of the following systems has a limit cycle:

(i) $\frac{dx}{dt} = y - x^3, \quad \frac{dy}{dt} = x - y - x^4 y;$

(ii) $\frac{dx}{dt} = y^2 - 2xy + y^4, \quad \frac{dy}{dt} = x^2 + y^2 + x^3 y^3;$

(iii) $\frac{dx}{dt} = x + xy^2, \quad \frac{dy}{dt} = x^2 + y^2.$

[10]

5. (a) Let T be the function $T : [0,1] \rightarrow [0,1]$ defined by

$$T(x) = \begin{cases} \frac{7}{4}x & 0 \leq x < \frac{1}{2} \\ \frac{7}{4}(1-x) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Determine the fixed points of periods one, two, and three.

[12]

(b) Determine the fixed points of periods one and two for the complex mapping

$$z_{n+1} = z_n^2 - 3.$$

Determine the stability of the fixed points of period one.

[8]

6. (a) Starting with an equilateral triangle (each side of length 1 unit) construct the inverted Koch snowflake up to stage two on graph paper. At each stage, each segment is $\frac{1}{3}$ the length of the previous segment, and each segment is replaced by four segments. Determine the area bounded by the true fractal and the fractal dimension.

[14]

(b) Prove that

$$D_1 = \lim_{l \rightarrow 0} \frac{\sum_{i=1}^{N} p_i \ln(p_i)}{-\ln(l)},$$

by applying L'Hopital's rule to the equation

$$D_q = \lim_{l \rightarrow 0} \frac{1}{1-q} \frac{\ln \sum_{i=1}^{N} p_i^q(l)}{-\ln l}.$$

[6]

7. (a) Find and classify the fixed points of period one of the Hénon map defined by

$$x_{n+1} = 1 - \frac{9}{5}x_n^2 + y_n \quad y_{n+1} = \frac{1}{5}x_n.$$

[8]

(b) Consider the complex iterative equation

$$E_{n+1} = A + BE_n \exp\left(i\left|E_n\right|^2\right).$$

Derive the inverse map and show that

$$\frac{d|A|^2}{d|E_S|^2} = 1 + B^2 + 2B\left(|E_S|^2 \sin|E_S|^2 - \cos|E_S|^2\right),$$

where E_S is a steady-state solution.

[12]

8. (a) A four-neuron discrete Hopfield network is required to store the following fundamental memories:

$$\mathbf{x}_1 = (1,1,1,1)^T, \quad \mathbf{x}_2 = (1,-1,1,-1)^T \quad \mathbf{x}_3 = (1,-1,-1,1)^T.$$

(i) Compute the synaptic weight matrix $\mathbf{W}$.

(ii) Use asynchronous updating to show that the three fundamental memories are stable.

(iii) Test the vector $(-1,-1,-1,1)^T$ on the Hopfield network.

Use your own set of random orders in (ii) and (iii).

[10]

(b) Derive a suitable Lyapunov function for the recurrent Hopfield network modeled using the differential equations

$$\dot{x} = -x + \left(\frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi x}{2}\right)\right) + \left(\frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi y}{2}\right)\right) + 6,$$

$$\dot{y} = -y + \left(\frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi x}{2}\right)\right) + 4\left(\frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi y}{2}\right)\right) + 10.$$

[10]

19.2 Dynamical Systems with Maple

Typically, students would be required to answer five out of 8 questions in three hours. The examination would take place in a computer laboratory with access to Maple.

1. (a) The radioactive decay of Polonium 218 to Bismuth 214 is given by

$$^{218}\text{Po} \rightarrow {}^{214}\text{Pb} \rightarrow {}^{214}\text{Bi},$$

where the first reaction rate is $k_1 = 0.5s^{-1}$, and the second reaction rate is $k_2 = 0.06s^{-1}$.

(i) Write down the differential equations representing this system. Solve the ODEs.

(ii) Determine the amount of each substance after 20 seconds given that the initial amount of ^{218}Po was one unit. Assume that the initial amounts of the other two substances was zero.

(iii) Plot solution curves against time for each substance.

(iv) Plot a trajectory in three-dimensional space.

[14]

(b) Plot the limit cycle of the system

$$\frac{dx}{dt} = y + 0.5x(1 - 0.5 - x^2 - y^2), \quad \frac{dy}{dt} = -x + 0.5y(1 - x^2 - y^2).$$

Find the approximate period of this limit cycle.

[6]

2. (a) Two solutes X and Y are mixed in a beaker. Their respective concentrations $x(t)$ and $y(t)$ satisfy the following differential equations:

$$\frac{dx}{dt} = x - xy - \mu x^2, \quad \frac{dy}{dt} = -y + xy - \mu y^2.$$

Find and classify the critical points for $\mu > 0$, and plot possible phase portraits showing the different types of qualitative behavior. Interpret the results in terms of the concentrations of solutes X and Y.

[14]

(b) Determine the Hamiltonian of the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^2.$$

Plot a phase portrait.

[6]

3. (a) For the system

$$\frac{dx}{dt} = \mu x + x^3, \quad \frac{dy}{dt} = -y$$

sketch phase portraits for $\mu < 0$, $\mu = 0$, and $\mu > 0$. Plot a bifurcation diagram.

[10]

(b) Plot a phase portrait and Poincaré section for the forced Duffing system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - 0.3y - x^3 + 0.39\cos(1.25t).$$

Describe the behavior of the system.

[10]

4. (a) Given that $f(x) = 3.5x(1-x)$,
   (i) plot the graphs of $f(x), f^2(x), f^3(x)$ and $f^4(x)$;
   (ii) approximate the fixed points of periods one, two, three, and four, if they exist;
   (iii) determine the stability of each point computed in part (ii).

[14]

(b) Use Maple to approximate the fixed points of periods one, two, and three for the complex mapping $z_{n+1} = z_n^2 + 2 + 3i$.

[6]

5. (a) Find and classify the fixed points of period one for the Hénon map

$$x_{n+1} = 1.5 + 0.2y_n - x_n^2, \quad y_{n+1} = x_n.$$

Find the approximate location of fixed points of period two if they exist. Plot a chaotic attractor using suitable initial conditions.

[14]

(b) Using the derivative method, compute the Lyapunov exponent of the logistic map $x_{n+1} = \mu x_n(1 - x_n)$, when $\mu = 3.9$.

[6]

6. (a) Edit the given program for plotting a bifurcation diagram for the logistic map (see Chapter 12) to plot a bifurcation diagram for the tent map.

[10]

(b) Write a program to plot a Julia set $J(0, 1.3)$, for the mapping $z_{n+1} = z_n^2 + 1.3i$.

[10]

7. (a) Given the complex mapping $E_{n+1} = A + BE_n e^{i|E_n|^2}$, determine the number and approximate location of fixed points of period one when $A = 3.2$ and $B = 0.3$.

[10]

(b) Edit the given program for producing a triangular Koch curve (see Chapter 13) to produce a square Koch curve. At each stage one segment is replaced by five segments and the scaling factor is $\frac{1}{3}$.

[10]

8. (a) A six-neuron discrete Hopfield network is required to store the following fundamental memories:

$$\mathbf{x}_1 = (1, 1, 1, 1, 1, 1)^T,$$

$$\mathbf{x}_2 = (1, -1, 1, -1, -1, 1)^T,$$

$$\mathbf{x}_3 = (1, -1, -1, 1, -1, 1)^T.$$

(i) Compute the synaptic weight matrix $\mathbf{W}$.

(ii) Use asynchronous updating to show that the three fundamental memories are stable.

(iii) Test the vector $(-1, -1, -1, 1, 1, 1)^T$ on the Hopfield network.

Use your own set of random orders in (ii) and (iii).

[10]

(b) Derive a suitable Lyapunov function for the recurrent Hopfield network modeled using the differential equations

$$\dot{x} = -x + 2\left(\frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi x}{2}\right)\right), \quad \dot{y} = -y + 2\left(\frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi y}{2}\right)\right).$$

Plot a vector field plot and Lyapunov function surface plot for $\gamma = 0.5$.

[10]